Existence of Ground State for the NLS on Star-like Graphs

A joint work in collaboration with D. Finco and D. Noja

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&

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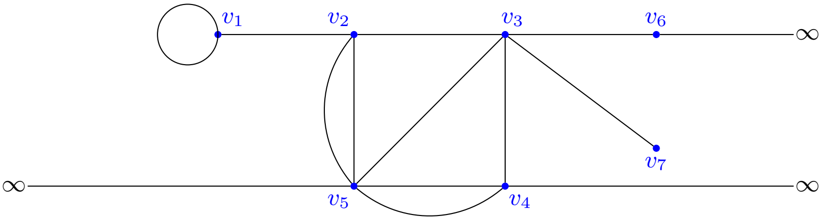
A Star-Like Graph



Metric graph: Each edge is associated either to a compact interval (if it is finite) or to $[0, +\infty)$ (if it is infinite)

$$E: \text{ denotes the set of edges of } \mathcal{G}$$
$$V: \text{ denotes the set of vertice of } \mathcal{G}$$

Assumption 1

$\mathcal{G}$ is a connected graph with a finite number of edges and vertices, and it is composed by at least one infinite edge (one half-line) attached to a compact core.

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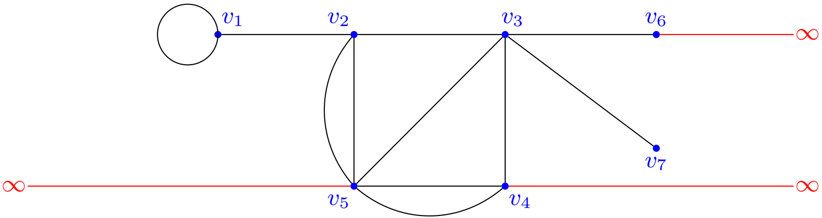

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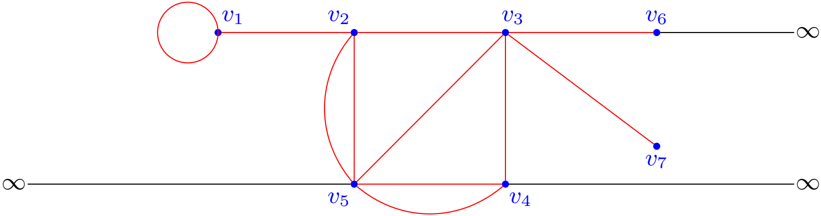

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Notation

Hilbert space: $\Psi \in L^2(\mathcal{G})$ means

$$\Psi = (\psi_1, \psi_2, ..., \psi_{|E|}) \qquad \psi_e \in L^2(I_e) \quad \forall e \in E$$

Sobolev spaces:

$$H^1(\mathcal{G}) := \left\{ \Psi \in L^2(\mathcal{G}) | \, \psi_e \in H^1(I_e) \, \forall e \in E \text{ and } \Psi \text{ is continuous in the vertices} \right\}$$

$$H^2(\mathcal{G}) := \left\{ \Psi \in H^1(\mathcal{G}) | \, \psi_e \in H^2(I_e) \, \forall e \in E \right\}$$

Scalar products and norms are defined in a natural way:

$$\|\Psi\|_{\mathcal{G}}^2 = \sum_{e \in E} \|\psi_e\|_{I_e}^2$$

The Nonlinear Schrödinger Equation

$$i\frac{d}{dt}\Psi = H\Psi - |\Psi|^{2\mu}\Psi \qquad 0 < \mu < 2$$

Linear term: H is a linear operator with δ-interaction in the vertices plus a potential

$$\mathcal{D}(H) := \left\{ \Psi \in H^2(\mathcal{G}) | \sum_{e \prec v} \partial_o \psi_e(v) = \alpha(v)\psi_e(v),\ \alpha(v) \in \mathbb{R},\ \forall v \in V \right\}.$$

$$H\Psi = -\Psi'' + W\Psi$$

Nonlinear term: Focusing powerlike nonlinearity, subcritical

Componentwise:

$$i\frac{d}{dt}\psi_e = -\frac{d^2}{dx_e^2}\psi_e + W_e\psi_e - |\psi_e|^{2\mu}\psi_e \qquad \forall e \in E$$

- $|E|$ scalar equations
- Coupled by the conditions in the vertices

The ground state for the NLS

Nonlinear energy functional: Defined on $H^1(\mathcal{G})$ as

$$E[\Psi] = \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v \in V} \alpha(v)|\Psi(v)|^2 - \frac{1}{\mu+1}\|\Psi\|_{2\mu+2}^{2\mu+2}$$

Ground state: Minimizer of $E[\Psi]$ at fixed mass $m = \|\Psi\|^2$

Problem: Under what conditions on $\mathcal{G}$, H, m the ground state does/does not exist

or equivalently

Under what conditions on $\mathcal{G}$, H, m the infimum

$$\inf\{E[\Psi] | \, \Psi \in H^1(\mathcal{G}), \, \|\Psi\|^2 = m\}$$

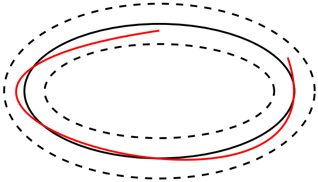
is/is not attained

Orbital Stability of Ground State

Let $\hat{\Psi}$ be a ground state, and consider the Cauchy problem:

$$\begin{cases} i\dfrac{d}{dt}\Psi = H\Psi - |\Psi|^{2\mu}\Psi \\ \Psi\Big|_{t=0} = \Psi_0 \end{cases} \tag{1}$$

$e^{i\omega t}\hat{\Psi}$, $\omega \in \mathbb{R}$, is the stationary solution of (1) with initial datum $\Psi_0 = \hat{\Psi}$.



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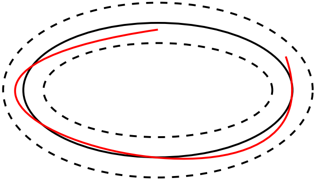


Theorem (Cazenave Lions '82)

Let $0 < \mu < 2$. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if

$$\|\Psi_0 - \hat{\Psi}\|_{H^1} \leq \delta(\varepsilon)$$

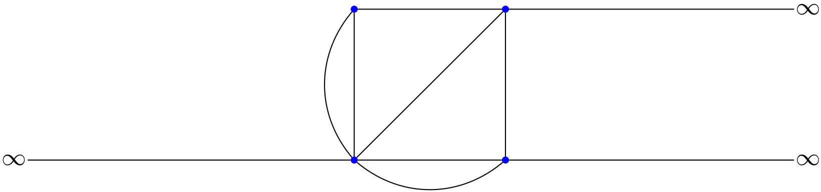
then the corresponding solution of (1) *is such that*

$$\sup_{t\in\mathbb{R}_+} \inf_{\theta\in\mathbb{R}} \|\Psi(t) - e^{i\theta}\hat{\Psi}\|_{H^1} \leq \varepsilon$$

Main Results: $W = 0$, $\alpha = 0$ [Adami-Serra-Tilli '14 '16]

- Take $W = 0$ and $\alpha(v) = 0$ for all $v \in V$
- Find topological and metric conditions on $\mathcal{G}$ that guarantee existence/non existence of the ground state

Condition H: From every point of the graph one can get to infinity through two disjoint paths



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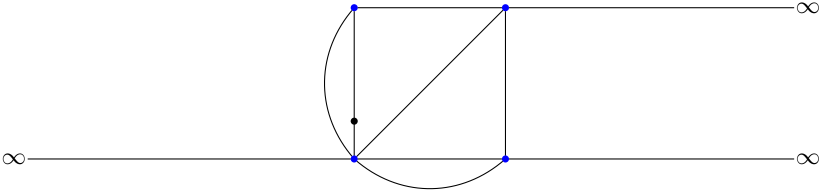

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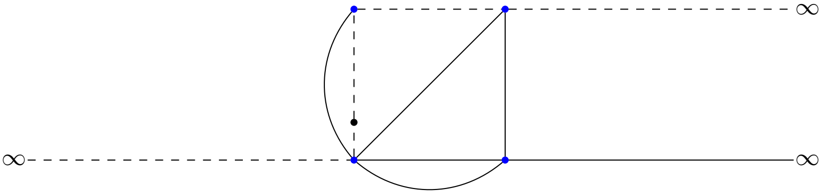

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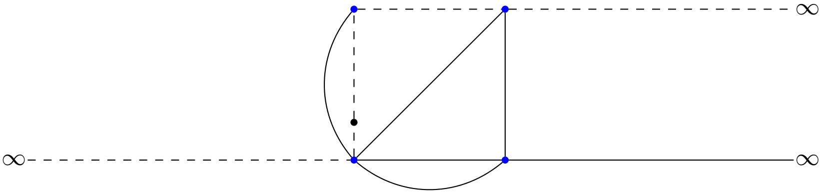
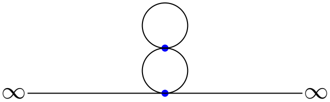
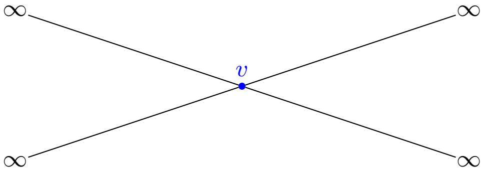


Adami-Serra-Tilli '14: If (H) is satisfied the ground state **does not** exist unless $\mathcal{G}$ is isometric to a bubble tower.



Main Results: Star-Graph, $W = 0$, $\alpha < 0$ [Adami, C.C., Finco, Noja '14]

- Consider a Star-Graph
- Take $W = 0$ and $\alpha(v) < 0$



Adami, C.C., Finco, Noja '14: There exists $m^* > 0$ such that for $0 < m < m^*$ the ground state exists.

Adami, Noja, Visciglia '13; Fukuizumi, Ohta, Ozawa '08: If $|E| = 2$ the ground state exists for any $m > 0$.

Adami, C.C., Finco, Noja '16: If $|E| \geq 3$ there exists $m^{**} > 0$ such that for $m > m^{**}$ the ground state **does not** exist.

Main Results: Generic Starlike-Graph [C.C., Finco, Noja prep. '16]

Assumption 2

$W = W_+ - W_-$ with $W_\pm \geq 0$, $W_+ \in L^1(\mathcal{G}) + L^\infty(\mathcal{G})$, and $W_- \in L^r(\mathcal{G})$ for some $r \in [1, 1 + 1/\mu]$.

Assumption 3

$\inf \sigma(H) := -E_0$, $E_0 > 0$ and it is an isolated eigenvalue.

Theorem

Let $0 < \mu < 2$. If Assumptions 1, 2, and 3 hold true then

$$-\infty < \inf\{E[\Psi] | \, \Psi \in H^1(\mathcal{G}), \, \|\Psi\|^2 = m\} \leq -E_0 m$$

for any $m > 0$. Moreover, there exists $m^* > 0$ such that for $0 < m < m^*$ the infimum is attained, i.e., the ground state exists.

Main Results: Generic Starlike-Graph [C.C., Finco, Noja prep. '16]

The inequality

$$\inf\left\{E[\Psi] \,|\, \Psi \in H^1(\mathcal{G}),\ \|\Psi\|^2 = m\right\} \leq -E_0 m$$

is a direct consequence of

$$E[\Psi] \leq \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v\in V} \alpha(v)|\Psi(v)|^2$$

for all $\Psi \in H^1(\mathcal{G})$ and of

$$-E_0 m = \inf\left\{\|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v\in V} \alpha(v)|\Psi(v)|^2 \,\middle|\, \Psi \in H^1(\mathcal{G}),\ \|\Psi\|^2 = m\right\}$$

Concentration-Compactness

For any $\Psi \in L^2(\mathcal{G})$ define the concentration function

$$\rho(\Psi, s) = \sup_{y\in\mathcal{G}} \|\Psi\|^2_{L^2(B_{\mathcal{G}}(y,s))} \,.$$

Let $\{\Psi_n\}_{n\in\mathbb{N}}$ be such that: $\Psi_n \in H^1(\mathcal{G})$,

$$\|\Psi_n\|^2 = m \qquad \sup_{n\in\mathbb{N}} \|\Psi'_n\| < C$$

Define the concentrated mass parameter τ as

$$\tau = \lim_{s\to\infty} \lim_{n\to\infty} \rho(\Psi_n, s) \,.$$

i) (Compactness) If $\tau = m$, at least one of the two following cases occurs:
   - i_1) (Convergence) There exists a function $\Psi \in H^1(\mathcal{G})$ such that $\Psi_n \to \Psi$ in $L^p(\mathcal{G})$ for all $2 \leq p \leq \infty$.
   - i_2) (Runaway) $\|\Psi_n\|_{L^p(B_{\mathcal{G}}(y,s))} \to 0$ for all $2 \leq p \leq \infty$, $y \in \mathcal{G}$, $s > 0$.

ii) (Vanishing) If $\tau = 0$, then $\Psi_n \to 0$ in $L^p(\mathcal{G})$ for all $2 < p \leq \infty$.

iii) (Dichotomy) If $0 < \tau < m$, then there exist two sequences $\{R_n\}_{n\in\mathbb{N}}$ and $\{S_n\}_{n\in\mathbb{N}}$ in $H^1(\mathcal{G})$ such that: $\|\Psi_n - R_n - S_n\| \to 0$,

$$\|R_n\|^2 \to \tau \qquad \|S_n\|^2 \to m - \tau$$

$$\mathrm{dist}(\mathrm{Supp}\ R_n, \mathrm{Supp}\ S_n) \to \infty$$

Concentration-Compactness

If Ψ_n is a minimizing sequence

- Vanishing and Dichotomy cannot occur
- If $i_2)$ (Runaway), then

$$\lim_{n\to\infty} E[\Psi_n] \geq -\gamma_\mu m^{1+\frac{2\mu}{2-\mu}}.$$

- $-\gamma_\mu m^{1+\frac{2\mu}{2-\mu}}$ is the energy of the ground state of mass m of the NLS on the real line

$$-\gamma_\mu m^{1+\frac{2\mu}{2-\mu}} = \inf_{\substack{\psi\in H^1(\mathbb{R}) \\ \|\psi\|^2_{L^2(\mathbb{R})}=m}} \left(\|\psi'\|^2_{L^2(\mathbb{R})} - \frac{1}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R})} \right)$$

Concentration-Compactness

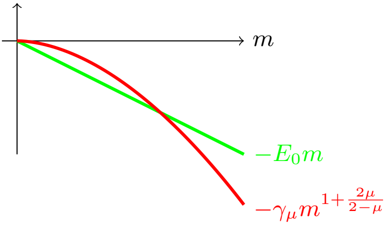
If Ψ_n is a minimizing sequence

- Vanishing and Dichotomy cannot occur
- If $i_2)$ (Runaway), then

$$\lim_{n\to\infty} E[\Psi_n] \geq -\gamma_\mu m^{1+\frac{2\mu}{2-\mu}}.$$

- But for m small enough

$$\inf\left\{E[\Psi] \,|\, \Psi \in H^1(\mathcal{G}),\ \|\Psi\|^2 = m\right\} \leq -E_0 m < -\gamma_\mu m^{1+\frac{2\mu}{2-\mu}}$$



- Indeed $m^* = (E_0/\gamma_\mu)^{\frac{1}{\mu}-\frac{1}{2}}$

Bifurcation Analysis

If E_0 is a simple eigenvalue one can use bifurcation theory to find a candidate. Consider the stationary equation

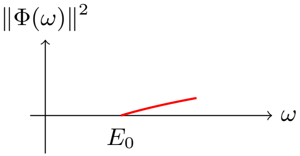
$$H\Phi - |\Phi|^{2\mu}\Phi = -\omega\Phi \qquad \Phi \in \mathcal{D}(H),\ \omega \in \mathbb{R}$$

Let $H\Phi_0 = -E_0\Phi_0$, with $\|\Phi_0\|^2 = 1$. Then for $\omega > E_0$ there exists a solution

$$\Phi(\omega) = a_*(\omega)\Phi_0 + \Theta_*(a_*(\omega), \omega)$$

such that

$$m(\omega) = \|\Phi(\omega)\|^2 = \left(\frac{\omega - E_0}{\|\Phi_0\|_{2\mu+2}^{2\mu+2}}\right)^{\frac{1}{\mu}} + o\left((\omega - E_0)^{\frac{1}{\mu}}\right)$$



$$E[\Phi(\omega(m))] = -E_0 m + o(m)$$

Remarks

- A sufficient condition to have an isolated eigenvalue is

$$\int_{\mathcal{G}} W\,dx + \sum_{v \in V} \alpha(v) < 0$$

- If $W = 0$ and $\alpha(v) \leq 0\ \forall v \in V$, and strictly negative for at least one vertex. Then $-E_0$ is a simple eigenvalue [Exner, Jex '12].
- For compact graphs with δ-vertices, simplicity of the spectrum can be achieved by small modification of edge lengths [Berkolaiko, Liu '16].
- The analysis can be extended in principle to the case in which $-E_0$ has multiplicity larger than one. One has to use bifurcation analysis in the degenerate case.
- We do not claim that $\Phi(\omega(m))$ is the ground state, even though we conjecture that this is true. This can be proved in the case of the Star-Graph with $W = 0$ and $\alpha(v) < 0$.
- Our result does not cover the case $W = 0$, $\alpha(v) = 0$, treated by Adami, Serra, Tilli. Since in this case there are not isolated eigenvalues, $\sigma(H) = \sigma_{ess}(H) = [0, +\infty)$.

Global Well-Posedness in $H^1(\mathcal{G})$

Theorem (Global Well-Posedness)

Let $0 < \mu < 2$. For any $\Psi_0 \in H^1(\mathcal{G})$, the Cauchy problem

$$\begin{cases} i\dfrac{d}{dt}\Psi = H\Psi - |\Psi|^{2\mu}\Psi \\ \Psi\Big|_{t=0} = \Psi_0 \end{cases} \qquad (2)$$

has a unique weak solution $\Psi \in C^0([0,\infty), H^1(\mathcal{G})) \cap C^1([0,\infty), H^1(\mathcal{G})^\star)$.

The proof uses the following conservation laws

Proposition (Conservation laws)

Let $\mu > 0$. For any weak solution $\Psi \in C^0([0,T), H^1(\mathcal{G})) \cap C^1([0,T), H^1(\mathcal{G})^\star)$ to the problem (2), the following conservation laws hold at any time t:

$$\|\Psi(t)\|^2 = \|\Psi_0\|^2, \qquad E[\Psi(t)] = E[\Psi_0].$$

Together with Local Well-posedness (proved by Banach fixed point theorem) and Gagliardo-Nirenberg inequalities

Gagliardo-Nirenberg Inequalities

Proposition

Let $\mathcal{G}$ be graph with a finite number of edges and vertices. Then if $p, q \in [2, +\infty]$, with $p \geq q$, and $\alpha = \frac{2}{2+q}(1 - q/p)$, there exists C such that

$$\|\Psi\|_p \ \leq \ C\|\Psi\|_{H^1}^{\alpha}\|\Psi\|_q^{1-\alpha},$$

for all $\Psi \in H^1(\mathcal{G})$.

If the $\mathcal{G}$ has one or more infinite edges one has the stronger inequality

$$\|\Psi\|_p \ \leq \ C\|\Psi'\|^{\alpha}\|\Psi\|_q^{1-\alpha},$$

See, e.g., [Mugnolo Springer '14, Adami-Serra-Tilli JFA '16].